Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 20:

Lesson/ Module: **Hamiltonian for the Electromagnetic Field**

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Learning Objectives:

From this module students may get to know about the following:

1. *Hamiltonian formalisms for fields and the canonical stress tensor.*
2. *The covariant generalization of the Poynting theorem, the law of conservation of energy.*
3. *Problems with the canonical stress tensor and their alleviation by symmetric stress tensor.*
4. *The covariant form for the angular momentum of the field.*

Hamiltonian and the Stress Tensors

20.1 Introduction

In this module we give a Hamiltonian description of field theory, in particular the electromagnetic field. As we know very well, in particle mechanics, the transition from the Lagrangian to the Hamiltonian formulation is made by first defining the canonical momentum variables by

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i}; \qquad i = 1, 2, \ldots n \tag{1}$$

L is the Lagrangian which is to be regarded as a function of the generalized coordinates q_i and generalized velocities $\dot{q}_i$ (which may be the ordinary coordinates and velocities) and perhaps an explicit function of time *t*. The Hamiltonian is then defined through

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{2}$$

Note that the Hamiltonian is to be regarded as a function of (q_i, p_i, t). It then follows from Hamilton's equation of motion that if the Lagrangian is not an explicit function of time, i.e., if $\partial L / \partial t = 0$, then $dH / dt = 0$, which means that the Hamiltonian is a constant of motion. This is an expression of the conservation of energy.

The Lagrangian approach to field theory was introduced in the last module. As we explained there, the approach to continuous fields closely parallels the techniques used for discrete point particles in mechanics. The finite number of generalized coordinates $q_i(t)$ and generalized velocities $\dot{q}_i(t)$, *i* = 1, 2, …, *n*, are replaced by an infinite number of degrees of freedom. Each point in space-time x^α corresponds to a finite number of values of the discrete index *i*. The value of the field at each point in space-time is a coordinate. Thus the generalized coordinate q_i is replaced by a continuous field $\phi_k(x)$ with discrete index *k* (= 1, 2,, *n*) and a continuous index, x^α. The generalized velocity $\dot{q}_i$ is replaced by a four-vector gradient, $\partial^\beta \phi_k(x)$. The Euler-Lagrange equations follow from the stationary property of the action integral with respect to variations $\delta\phi_k$ and $\delta(\partial^\beta \phi_k)$ around the physical values. We thus have the following correspondences:

$$\begin{aligned} i &\to x^\alpha, k \\ q_i &\to \phi_k(x) \\ \dot{q}_i &\to \partial^\alpha \phi_k(x) \end{aligned} \tag{3}$$

$$L = \sum_i L_i(q_i, \dot{q}_i) \to \int \tilde{L}(\phi_k, \partial^\alpha \phi_k) d^3x \tag{4}$$

The Euler-Lagrange equations take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \left(\frac{\partial L}{\partial q_i}\right) \rightarrow \partial^\beta \frac{\partial \tilde{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \tilde{L}}{\partial \phi_k} \tag{5}$$

20.2 The canonical Stress Tensor

Similarly, we need to construct a *Hamiltonian density* $\tilde{H}$ whose volume integral over three-dimensional space, *H*, can be interpreted as energy. The Lorentz transformation property of $\tilde{H}$ can be guessed as follows. Since the energy of a particle is the time component of a four-vector (the energy-momentum four-vector), the Hamiltonian should transform in the same way. Since $H = \int \tilde{H} d^3x$, and the invariant four-volume element is $d^4x = d^3x dx_0 = cd^3x dt$, it is necessary that the Hamiltonian density transform as the time-time component of a second rank four-tensor. From the Lagrangian density $\tilde{L}(\phi_k(x), \partial^\alpha \phi_k(x))$, we construct the Hamiltonian density in analogy with equation (2) as

$$\tilde{H} = \sum_k \frac{\partial \tilde{L}}{\partial\left(\frac{\partial \phi_k}{\partial t}\right)} \frac{\partial \phi_k}{\partial t} - \tilde{L} \tag{6}$$

The first factor in the sum, $\frac{\partial \tilde{L}}{\partial\left(\frac{\partial \phi_k}{\partial t}\right)}$, is the field momentum canonically conjugate to $\phi_k(x)$ and $\frac{\partial \phi_k}{\partial t}$ is equivalent to the velocity $\dot{q}_i$. The inferred Lorentz transformation property of $\tilde{H}$ suggest that the *covariant generalization* of the Hamiltonian density is the *canonical stress tensor*:

$$T^{\alpha\beta} = \sum_k \frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta} \tilde{L} \tag{7}$$

For the electromagnetic field, the field variables are A^α and $\partial^\beta A^\alpha$, so we make the replacements

$$\phi_k \rightarrow A^\alpha, \partial^\beta \phi_k \rightarrow \partial^\beta A^\alpha$$

so that

$$T^{\alpha\beta} = \frac{\partial \tilde{L}_{em}}{\partial(\partial_\alpha A^\lambda)} \partial^\beta A^\lambda - g^{\alpha\beta} \tilde{L}_{em} \tag{8}$$

For the *free* electromagnetic field the Lagrangian is

$$\tilde{L}_{free} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \tag{9}$$

so that

$$\frac{\partial \tilde{L}_{free}}{\partial(\partial_\alpha A^\lambda)} = -\frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_\alpha A^\lambda)} (F^{\mu\nu} F_{\mu\nu}) = -\frac{1}{4\mu_0} [\frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A^\lambda)} F_{\mu\nu} + \frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A^\lambda)} F^{\mu\nu}]$$

Let us look at the first term. Using the relation between the field and the potentials

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

we write

$$\frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A^\lambda)} F_{\mu\nu} = \frac{\partial(\partial^\mu A^\nu - \partial^\nu A^\mu)}{\partial(\partial_\alpha A^\lambda)} F_{\mu\nu} = \frac{\partial(g^{\mu\sigma}\partial_\sigma A^\nu - g^{\nu\sigma}\partial_\sigma A^\mu)}{\partial(\partial_\alpha A^\lambda)} F_{\mu\nu}$$
$$= (g^{\mu\sigma}\delta^\alpha{}_\sigma \delta^\nu{}_\lambda - g^{\nu\sigma}\delta^\alpha{}_\sigma \delta^\mu{}_\lambda) F_{\mu\nu} = g^{\mu\alpha} F_{\mu\lambda} - g^{\nu\alpha} F_{\lambda\nu} = 2g^{\mu\alpha} F_{\mu\lambda}$$

Proceeding in the same fashion the second term also produces the same result, so that

$$\frac{\partial \tilde{L}_{free}}{\partial(\partial_\alpha A^\lambda)} = -\frac{1}{4\mu_0}(4g^{\mu\alpha} F_{\mu\lambda}) = -\frac{1}{\mu_0} g^{\mu\alpha} F_{\mu\lambda}$$

and

$$T^{\alpha\beta} = -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \tilde{L}_{free} \tag{10}$$

To elucidate the meaning of this tensor, let us look at its specific components. From the explicit forms of $F^{\alpha\beta}$ and $F_{\alpha\beta}$:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{11}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{12}$$

$$\tilde{L}_{free} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2\mu_0}(E^2/c^2 - B^2) \qquad (13)$$

From equation (10)

$$T^{00} = -\frac{1}{\mu_0} g^{0\mu} F_{\mu\lambda}\partial^0 A^\lambda - \frac{1}{2\mu_0}(E^2 - B^2) g^{00} = -\frac{1}{\mu_0} g^{00} F_{0\lambda}\partial^0 A^\lambda - \frac{1}{2\mu_0}(E^2/c^2 - B^2)$$
$$= -\frac{1}{\mu_0 c^2}(E_x \frac{\partial}{\partial t} A_x + E_y \frac{\partial}{\partial t} A_y + E_z \frac{\partial}{\partial t} A_z) - \frac{1}{2\mu_0}(E^2 - B^2) = -\frac{1}{\mu_0 c^2} \vec{E}.\frac{\partial}{\partial t}\vec{A} - \frac{1}{2\mu_0}(E^2/c^2 - B^2)$$

Now from the definition of the electric field in terms of the potentials

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \rightarrow -\frac{\partial \vec{A}}{\partial t} = \vec{E} + \vec{\nabla}\Phi \qquad (14)$$

On using this relation and the fact that for free electromagnetic field, $\vec{\nabla}.\vec{E} = 0$, we have

$$T^{00} = \frac{1}{\mu_0 c^2}\vec{E}.(\vec{E} + \vec{\nabla}\Phi) - \frac{1}{2\mu_0}(E^2/c^2 - B^2) = \frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0) + \varepsilon_0 \vec{\nabla}.(\Phi\vec{E}) \qquad (15)$$

Since $T^{\alpha\beta}$ was postulated as the covariant generalization of the Hamiltonian, we expect T^{00} to represent energy density. The above expression does contain the desired term $\frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)$ which is the field energy density. But an additional term is also present, viz., $\vec{\nabla}.(\Phi\vec{E})$. If we suppose that the fields are localized in some finite region of space (and because of the finite velocity of propagation, they always are), then on integration over all space the volume integral of this last term is converted into a surface integral which vanishes:

$$\int_V \vec{\nabla}.(\Phi\vec{E}) d^3x = \oint_S (\Phi\vec{E}).\hat{n} da = 0$$

Thus

$$\int T^{00} d^3x = \frac{1}{2\pi}\int (\varepsilon_0 E^2 + B^2/\mu_0) d^3x = W_{field} \qquad (16)$$

This, of course, is the usual expression for the energy of the electromagnetic field.

Now let us look at the T^{0i} component of the canonical stress tensor:

$$T^{0i} = -\frac{1}{\mu_0} g^{0\mu} F_{\mu\lambda}\partial^i A^\lambda = -\frac{1}{\mu_0} F_{0\lambda}\partial^i A^\lambda$$
$$= \frac{1}{\mu_0 c}(E_1 \frac{\partial}{\partial x_i} A_1 + E_2 \frac{\partial}{\partial x_i} A_2 + E_3 \frac{\partial}{\partial x_i} A_3) = \frac{1}{\mu_0 c} E_j \frac{\partial}{\partial x_i} A_j$$

Further

$$(\vec{E}\times\vec{B})_i = [\vec{E}\times(\vec{\nabla}\times\vec{A})]_i = \varepsilon_{ijk} E_j \varepsilon_{klm} \frac{\partial}{\partial x_l} A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \frac{\partial}{\partial x_l} A_m$$
$$= E_j \frac{\partial}{\partial x_i} A_j - E_j \frac{\partial}{\partial x_j} A_i$$

Hence

$$E_j \frac{\partial}{\partial x_i} A_j = (\vec{E}\times\vec{B})_i + E_j \frac{\partial}{\partial x_j} A_i = (\vec{E}\times\vec{B})_i + \vec{E}.\vec{\nabla} A_i = (\vec{E}\times\vec{B})_i + \vec{\nabla}.(\vec{E}A_i)$$

On substituting this into the expression for T^{0i}, we have

$$T^{0i} = \frac{1}{\mu_0 c}(\vec{E}\times\vec{B})_i + \frac{1}{\mu_0 c}\vec{\nabla}.(\vec{E}A_i) \qquad (17)$$

One would have expected T^{0i} to represent something like momentum density which it does, except for the additional term $\frac{1}{\mu_0 c}\vec{\nabla}.(\vec{E}A_i)$. Once again the additional term is a divergence and on integration over all space changes into a surface integral via Gauss theorem and vanishes so that

$$\int T^{0i} d^3x = \frac{1}{\mu_0 c}\int (\vec{E}\times\vec{B}) d^3x = c\vec{P}^i{}_{field} \qquad (18)$$

This is the usual expression for the electromagnetic field momentum.

In a similar calculation we also obtain

$$T^{i0} = \frac{1}{\mu_0 c}(\vec{E}\times\vec{B})_i + \frac{1}{\mu_0 c}[(\vec{\nabla}\times\Phi\vec{B})_i - \frac{\partial}{\partial x_0}(\Phi E_i)] \qquad (19)$$

- Evidently the canonical stress tensor $T^{\alpha\beta}$ is not symmetric. This is clear from the definition [equation (10)] itself, since it contains the term $\partial^\beta A^\lambda$ but not its counterpart $\partial^\lambda A^\beta$.

20.2.1 Poynting Theorem

The connection of the time-time and time-space components of $T^{\alpha\beta}$ with the energy and momentum density of the field suggests that there is a covariant generalization of the Poynting theorem or the differential conservation law of energy, viz.,

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{20}$$

We prove this relation for the general case described by the tensor (7) and Euler-Lagrange equations (5). Consider

$$\partial_\alpha T^{\alpha\beta} = \sum_k \partial_\alpha [\frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k] - \partial^\beta \tilde{L}$$

$$= \sum_k \{\partial_\alpha [\frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)}] \partial^\beta \phi_k + \frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} \partial_\alpha \partial^\beta \phi_k\} - \partial^\beta \tilde{L}$$

By means of the Euler-Lagrange equation (5)

$$\partial^\beta \frac{\partial \tilde{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \tilde{L}}{\partial \phi_k} \rightarrow \partial_\alpha \frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} = \frac{\partial \tilde{L}}{\partial \phi_k}$$

the first term above can be transformed and we obtain

$$\partial_\alpha T^{\alpha\beta} = \sum_k [\frac{\partial \tilde{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} \partial_\alpha \partial^\beta \phi_k] - \partial^\beta \tilde{L}$$

The Lagrangian density is a function of ϕ_k and $\partial^\alpha \phi_k$ which are themselves functions of four-vector *x*, so by the chain rule of partial differentiation

$$\partial^\beta \tilde{L}(\phi_k, \partial^\alpha \phi_k) = \sum_k [\frac{\partial \tilde{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \tilde{L}}{\partial(\partial_\alpha \phi_k)} \partial_\alpha \partial^\beta \phi_k]$$

Hence

$$\partial_\alpha T^{\alpha\beta} = \partial^\beta \tilde{L} - \partial^\beta \tilde{L} = 0$$

The conservation law or continuity equation (18) yields the laws of conservation of energy and momentum. On integrating the equation over all space we obtain

$$0 = \int \partial_\alpha T^{\alpha\beta} d^3x = \partial_0 \int T^{0\beta} d^3x + \int \partial_i T^{i\beta} d^3x$$

The second integral, being a divergence, can be converted into a surface integral by use of Gauss theorem

$$\int_V \partial_i T^{i\beta} d^3x = \oint_S T^{i\beta} n_i da$$

If the fields are localized, which they are, as we have seen, the contribution of this integral vanishes and we have

$$\partial_0 \int T^{0\beta} d^3x = 0$$

Since T^{00} is the energy and T^{0i} the momentum of the field, we have

$$\frac{d}{dt} W_{field} = 0; \quad \frac{d}{dt} \vec{P}_{field} = 0 \tag{21}$$

20.3 The Symmetric Stress Tensor

The canonical stress tensor, $T^{\alpha\beta}$, while adequate so far, has a certain number of deficiencies.

- Evidently the canonical stress tensor $T^{\alpha\beta}$ is not symmetric. This is clear from the definition [equation (8)] itself, since it contains the term $\partial^\beta A^\lambda$ but not its counterpart $\partial^\lambda A^\beta$.
- Though the integrals of T^{00} and T^{0i} represent the energy and (c times) momentum respectively, T^{00} and T^{0i} differ from the expressions for the energy and momentum densities.
- Further, the canonical stress tensor involves the potentials and the expression therefore is not gauge invariant.
- Finally we add that the trace of $T^{\alpha\beta}$ is not zero, a requirement that comes from the zero mass of the photons.

To obtain a symmetric, traceless, gauge-invariant stress tensor $\Theta^{\alpha\beta}$ from the canonical stress tensor, $T^{\alpha\beta}$, we proceed as follows. From the definition of the field tensor, we have

$$F^{\beta\lambda} = \partial^\beta A^\lambda - \partial^\lambda A^\beta \Rightarrow \partial^\beta A^\lambda = \partial^\lambda A^\beta - F^{\beta\lambda}.$$

Substitute this into expression for $T^{\alpha\beta}$, equation (8), so that

$$T^{\alpha\beta} = \frac{1}{\mu_0}[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}] - \frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta \tag{22}$$

The first two terms in equation (22) are gauge invariant (as they do not involve the potentials directly) and symmetric in α and β. With the help of the Maxwell equations for the source-free case

$$\vec{\nabla}.\vec{E} = 0; \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$$

or

$$\partial_\lambda F^{\lambda\alpha} = 0$$

the last term can be recast as

$$\begin{aligned} T_D{}^{\alpha\beta} &\equiv -\frac{1}{\mu_0} g^{\alpha\mu} F_{\mu\lambda}\partial^\lambda A^\beta = -\frac{1}{\mu_0} F^{\alpha}{}_{\lambda}\partial^\lambda A^\beta = -\frac{1}{\mu_0} F^{\alpha\lambda}\partial_\lambda A^\beta \\ &= \frac{1}{\mu_0} F^{\lambda\alpha}\partial_\lambda A^\beta = \frac{1}{\mu_0}[F^{\lambda\alpha}\partial_\lambda A^\beta + A^\beta \partial_\lambda F^{\lambda\alpha}] \qquad (23) \\ &= \frac{1}{\mu_0}\partial_\lambda [F^{\lambda\alpha} A^\beta] \end{aligned}$$

Now

$$\partial_\alpha T_D{}^{\alpha\beta} = \frac{1}{\mu_0}\partial_\alpha \partial_\lambda [F^{\lambda\alpha} A^\beta] = 0 \qquad (24)$$

since $\partial_\alpha \partial_\lambda$ is symmetric while $F^{\lambda\alpha}$ is antisymmetric. Further

$$T_D{}^{0\beta} = \frac{1}{\mu_0}\partial_\lambda [F^{\lambda 0} A^\beta] = \frac{1}{\mu_0}\frac{\partial}{\partial x_i}[F^{i0} A^\beta] = \frac{1}{\mu_0}\vec{\nabla}.(A^\beta \vec{E})$$

Hence

$$\int_V T_D{}^{0\beta} d^3x = \int_V \frac{1}{\mu_0}\vec{\nabla}.(A^\beta \vec{E}) d^3x = \oint_S \frac{1}{\mu_0}\hat{n}.(A^\beta \vec{E}) da = 0 \qquad (25)$$

for localized fields.

Since $T^{\alpha\beta}$ and $T_D{}^{\alpha\beta}$ both satisfy differential conservation laws [equations (20) and (24) respectively], so does their difference $T^{\alpha\beta} - T_D{}^{\alpha\beta}$. Similarly the integral relations (16) and (18) are also valid for the difference. We are therefore free to *define the symmetric stress tensor* $\Theta^{\alpha\beta}$

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - T_D{}^{\alpha\beta} = \frac{1}{\mu_0}[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}] \qquad (26)$$

with the property

$$\partial_\alpha \Theta^{\alpha\beta} = 0\,. \qquad (27)$$

Using the explicit expressions for $F^{\alpha\beta}$ and $F_{\alpha\beta}$, equations (11) and (12) respectively, we obtain

$$\Theta^{00} = \frac{1}{\mu_0}[g^{0\mu}F_{\mu\lambda}F^{\lambda 0} + \frac{1}{4}g^{00}F_{\mu\nu}F^{\mu\nu}] = \frac{1}{\mu_0}[F_{0\lambda}F^{\lambda 0} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}]$$

$$= \frac{1}{\mu_0}[E^2/c^2 + \frac{1}{4}2(B^2 - E^2/c^2)]$$

or

$$\Theta^{00} = \frac{1}{2}(\varepsilon_0 E^2 + B^2/\mu_0)\,; \tag{28}$$

Similarly

$$\Theta^{0i} = \Theta^{i0} = \frac{1}{c\mu_0}(\vec{E}\times\vec{B})_i \tag{29}$$

$$\Theta^{ij} = -[\varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\delta_{ij}(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2)] \tag{30}$$

Here the Latin indices *i* and *j*, as throughout, refer to Cartesian components in 3-space. Recall that Θ^{00} is the energy density of the electromagnetic field, *u*; Θ^{0i} is *c* times the momentum density of the electromagnetic field, $\vec{g}$ and Θ^{ij} is the negative of the *Maxwell stress tensor* $M_{ij}^{(M)}$, which is related to the flow of momentum across a surface. The tensor $\Theta^{\alpha\beta}$ can be written in a schematic *block matrix* form as

$$\Theta^{\alpha\beta} = \begin{pmatrix} u & c\vec{g} \\ c\vec{g} & -T_{ij}^{(M)} \end{pmatrix}$$

Here *u* is a single entry, $c\vec{g}$ is a row vector, $c\vec{g}^T$ is the transposed column vector and $T_{ij}^{(M)}$ is a 3×3 matrix. The various other, covariant and mixed, forms of the stress tensor are

$$\Theta_{\alpha\beta} = g_{\alpha\gamma}\Theta^{\gamma\delta}g_{\delta\beta} = \begin{pmatrix} u & -c\vec{g} \\ -c\vec{g} & -T_{ij}^{(M)} \end{pmatrix}$$

$$\Theta^{\alpha}{}_{\beta} = g^{\alpha\gamma}\Theta_{\gamma\beta} = \begin{pmatrix} u & -c\vec{g} \\ c\vec{g} & T_{ij}^{(M)} \end{pmatrix}$$

$$\Theta_{\alpha}{}^{\beta} = g_{\alpha\gamma}\Theta^{\gamma\beta} = \begin{pmatrix} u & c\vec{g} \\ -c\vec{g} & T_{ij}^{(M)} \end{pmatrix}$$

The differential conservation law [equation (27)]:

$$\partial_\alpha \Theta^{\alpha\beta} = 0$$

embodies Poynting's theorem and conservation of momentum for free fields. For example, for $\beta = 0$ we have

$$0 = \partial_\alpha \Theta^{\alpha 0} = \frac{1}{c}\left(\frac{\partial u}{\partial t} + \vec{\nabla}.\vec{S}\right),$$

where $\vec{S} = c^2 \vec{g}$ is the Pointing vector. This is the differential form of the law of conservation of energy in the absence of sources: the rate at which the energy in a volume increases is the negative of the rate at which it flows out of the volume. Similarly for $\beta = i$,

$$0 = \partial_\alpha \Theta^{\alpha i} = \frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}^{(M)}$$

which is the differential form of the law of conservation of momentum for the field in the absence of the sources.

20.4 The Angular Momentum

In particle mechanics the angular momentum of a system of particles can be defined through the antisymmetric tensor

$$M_{ij} = \sum_{all\ particles} [x_i p_j - x_j p_i]$$

This tensor has three independent components which together form the angular momentum $\vec{L} = \vec{r} \times \vec{p}$. Extending this idea to four dimensional space, we define an antisymmetric tensor $M^{\alpha\beta}$ by

$$M^{\alpha\beta} = \sum_{all\ particles} [x^\alpha p^\beta - x^\beta p^\alpha]$$

This has six independent components of which the three components corresponding to the space part M_{ij} represent the angular momentum of the system, as before. Conservation of angular momentum is expressed by the statement M_{ij} = constant. Thus we conjecture that the full conservation law is $M^{\alpha\beta}$ = constant. Then the constancy of the three space-time components of $M^{\alpha\beta}$ is expressed as

$$M^{i0} = const \Rightarrow \sum [x^i \frac{W}{c} - ctp^i] = const$$

If we divide by the total energy, $\sum W$, we get

$$\frac{\sum x^i W}{\sum W} = \frac{\sum c^2 t p^i}{\sum W} + const$$

The sum on the left hand side is the position of the centre of mass of the system

$$\vec{r}_{cm} = \frac{\sum \gamma m \vec{x}}{\sum \gamma m}$$

while that on the right is *t* times the centre of mass velocity. Thus we obtain

$$\vec{v}_{cm} = \frac{\sum \gamma m \vec{v}}{\sum \gamma m}$$

a very reasonable result.

To get the equivalent result for the electromagnetic field, we define the field angular momentum density by

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^\gamma - \Theta^{\alpha\gamma} x^\beta$$

which is the covariant generalization of the angular momentum density in three-space. Then

$$\begin{aligned}
&\partial_\alpha M^{\alpha\beta\gamma} = \partial_\alpha (\Theta^{\alpha\beta} x^\gamma - \Theta^{\alpha\gamma} x^\beta) \\
&= \partial_\alpha [\Theta^{\alpha\beta}] x^\gamma + [\Theta^{\alpha\beta}] \partial_\alpha x^\gamma - \partial_\alpha [\Theta^{\alpha\gamma}] x^\beta - [\Theta^{\alpha\gamma}] \partial_\alpha x^\beta \\
&= [\Theta^{\alpha\beta}] \partial_\alpha x^\gamma - [\Theta^{\alpha\gamma}] \partial_\alpha x^\beta = \Theta^{\gamma\beta} - \Theta^{\beta\gamma} = 0
\end{aligned}$$

This is the differential form of the law of conservation of angular momentum for the field in the absence of sources.

- The symmetry of $\Theta^{\alpha\beta}$ is quite crucial to this proof of conservation of angular momentum. Had we used the canonical stress tensor $T^{\alpha\beta}$ instead, conservation of angular momentum would not have been possible.

Summary

1. *Hamiltonian formalism for fields is described. Canonical stress tensor is introduced. The Hamiltonian density is a component of a second order tensor represented by the canonical stress tensor.*
2. *The formalism is then developed for the electromagnetic field in particular.*
3. *The covariant generalization of the Poynting theorem, the law of conservation of energy, is given. The covariant generalization includes the law of conservation of energy and of momentum.*
4. *Problems with the canonical stress tensor are discussed and the symmetric stress tensor introduced which eliminates these problems.*
5. *The covariant form for the angular momentum of the field is given that needs the introduction of a third rank tensor. The law of conservation of angular momentum is obtained in terms of this third rank tensor.*